

1.5 PERCENTAGE CHANGE – FINANCIAL APPLICATIONS

In problems with

Interest rates $r\%$

Population growth $r\%$, etc

where

some initial quantity (**present value PV**)

increases by $r\%$ (per year, per month or per any period)

the **future value FV** after n periods is given by

$$FV = PV \left(1 + \frac{r}{100} \right)^n$$

If the present value PV **decreases** by $r\%$ per period, the formula takes the form

$$FV = PV \left(1 - \frac{r}{100} \right)^n$$

EXAMPLE 1

- (a) An amount of 2000 euros is **invested** at 8% per year.
Find the value of the investment after 10 years?
- (b) An amount of 2000 euros is **depreciated** by 8% every year.
Find the amount returned after 10 years?

Solution

For both problems $PV=2000$, $r=8$, $n=10$

$$(a) FV = PV \left(1 + \frac{r}{100} \right)^n = 2000 \left(1 + \frac{8}{100} \right)^{10} = 2000(1.08)^{10} \cong 4317.85$$

$$(b) FV = PV \left(1 - \frac{r}{100} \right)^n = 2000 \left(1 - \frac{8}{100} \right)^{10} = 2000(0.92)^{10} \cong 868.78$$

Remark: In financial mathematics we usually answer in 2dp.

Notice: Using GDC (Casio) for Exercise 1

MENU - Financial - F2: [Compound Interest]

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n = 10
I% = 8
PV = -2000 (use "-" because we pay)
PMT=0
FV = ?
P/Y=1
C/Y=1

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Press F5: [FV] and you obtain FV=4317.85

For question (b), set I% = -8. You obtain FV=868.78

In fact, this is a geometric sequence with common ratio $R=1+\frac{r}{100}$

Explanation for the common ratio R

Suppose we invest an amount PV for r% per year.

In our example: PV = 2000 r = 8%

	In general	In our example
Present Value	PV	2000
Interest	$PV \times \frac{r}{100}$	$2000 \times \frac{8}{100}$
After 1 year	$PV + PV \frac{r}{100}$ $= PV \left(1 + \frac{r}{100} \right)$	$2000 + 2000 \times \frac{8}{100}$ $2000 \left(1 + \frac{8}{100} \right)$

That is, if an amount increases by r%, we multiply by $R=1+\frac{r}{100}$ After two years we multiply again by R. Thus $FV = PV \left(1 + \frac{r}{100} \right)^2$ Thus, after n years: $FV = PV \left(1 + \frac{r}{100} \right)^n$

The rate $r\%$ is translated into a common ratio as follows:

$r\%$	(increasing ratio) $R=1+\frac{r}{100}$	(decreasing ratio) $R=1-\frac{r}{100}$
12%	$R = 1.12$	$R = 0.88$
20%	$R = 1.20$	$R = 0.80$
5%	$R = 1.05$	$R = 0.95$

Mind the following slight difference between the formulas

$$FV = PV \left(1 + \frac{r}{100} \right)^n \quad \text{and} \quad u_n = u_1 r^{n-1}$$

Is the initial amount given in “year 1” or “today”?

Suppose that the interest rate is 12%, so that $R=1.12$

PROBLEM 1	PROBLEM 2
<u>Amount in year 1:</u> $u_1=1000$. Find the amount in year n. Then - In year 2 $u_2=1000 \times (1.12)$ - In year 3 $u_3=1000 \times (1.12)^2$ - In year n $u_n=1000 \times (1.12)^{n-1}$	<u>Present value:</u> $PV=1000$. Find the amount after n years. Then - After 1 year $1000 \times (1.12)$ - After 2 years $1000 \times (1.12)^2$ - After n years $FV=1000 \times (1.12)^n$

(Mind that the exponent in Problem 2 is n and not $n-1$)

In both cases the growth is **exponential** (geometric sequence)

EXAMPLE 2

At the beginning of 2024, we invest an amount of 5000€, with an interest rate of 8%. Find the value of the investment at the beginning of 2031.

Solution

Here $n=7$ (after 7 years). So $FV=5000 \left(1 + \frac{8}{100} \right)^7 = 8569.12$

Sometimes the question is about the number of years to reach future value.

EXAMPLE 3

An amount of 2000 euros is invested at 8% per year. The investment exceeds 5000 after n complete years. Find n .

Solution

$$FV = 2000 \left(1 + \frac{8}{100} \right)^n > 5000$$

Method A (trial and error): We check several values for n by GDC and realize that

$$\text{for } n=11 \quad FV = 4663.27$$

$$\text{for } n=12 \quad FV = 5036.34$$

Therefore, $n = 12$

Method B (by using SolveN in the GDC):

$$5000 = 2000 \left(1 + \frac{8}{100} \right)^n$$

The solution is $n \cong 11.9$. Thus $n = 12$

Method C (by using GDC-Financial mode):

$I\% = 8$
$PV = -2000$
$PMT = 0$
$FV = 5000$
$P/Y = 1$
$C/Y = 1$

Press F1: [n] and you obtain $n=11.9$

Since we are looking for complete years we accept the first integer above 11.9, that is $n = 12$

EXAMPLE 4

The current population of a city is 800 000.

The population increases by 5.2% every year.

- (a) Find the population of the city after 7 years;
- (b) Find the population of the city 7 years ago;
- (c) The population doubles after n full years. Find n .

Solution

- (a) We have an exponential growth with $PV=800,000$ and $r=8\%$

The population of the city after 7 years is

$$FV=800,000\left(1+\frac{5.2}{100}\right)^7 \cong 1,140,775$$

- (b) The formula works for the past as well!

The population of the city 7 years ago was

$$FV=800,000\left(1+\frac{5.2}{100}\right)^{-7} \cong 561,022$$

[short explanation:

for the future we multiply by 1.052 every year

for the past we divide by 1.052 every year, or otherwise we

multiply by $\frac{1}{1.052}=1.052^{-1}$ every year]

- (c) The future value is $FV=2 \times 800\,000 = 1\,600\,000$

Hence

$$800\,000\left(1+\frac{5.2}{100}\right)^n = 1\,600\,000$$

By using a GDC we find $n=13.7$

Therefore, the population doubles after 14 complete years.

Notice

You may also use the financial mode of the GDC to obtain the results.

♦ INTEREST COMPOUNDED IN k TIME PERIODS

Suppose that an initial amount $PV=1000\text{€}$ is invested with an interest rate 12% per year. We have seen what happens if the interest is compounded yearly.

However, the interest may be compounded in k periods per year:

Semiannually (half-yearly): $k=2$

Quarterly: $k=4$

Monthly: $k=12$

For example, if the interest is compounded twice per year (semiannually), the interest rate for the 6-month period is 6% and

$$\text{Amount after 1 year} = 1000 \times (1.06)^2 \quad (1 \text{ year} = 2 \text{ time periods})$$

In general, the FV formula takes the form

$$FV = PV \left(1 + \frac{r}{100k} \right)^{kn}$$

For our example of $PV = 1000$:

Compounded		After 5 years	
yearly	($k=1$)	$FV=1000 \left(1 + \frac{12}{100} \right)^5$	= 1762
half-yearly	($k=2$)	$FV=1000 \left(1 + \frac{12}{100 \times 2} \right)^{2 \times 5}$	= 1791
quarterly	($k=4$)	$FV=1000 \left(1 + \frac{12}{100 \times 4} \right)^{4 \times 5}$	= 1806
monthly	($k=12$)	$FV=1000 \left(1 + \frac{12}{100 \times 12} \right)^{12 \times 5}$	= 1817

NOTICE for GDC-Financial mode

C/Y stands for the number k of periods per year. Check FV for

$n = 5$	$PMT=0$
$I\% = 12$	$P/Y = 1$
$PV = -1000$	$C/Y = 1, 2, 4, 12 \text{ respectively}$

♦ INVESTMENT WITH EXTRA REGULAR PAYMENTS

PV is invested with an annual interest rate $r\%$.

An extra payment (PMT) is added at the end of each year

In this case we strongly suggest GDC-Financial mode.

EXAMPLE 5

An amount of 1000€ is invested (at the beginning of some year) with an interest rate 12% compounded yearly. An extra payment of 300€ is added at the end of each year. Find the value of the investment after 7 years.

By using GDC-Financial mode

$n = 7$	$PMT = -300$ ["-" as we pay]
$I\% = 12$	$P/Y = 1$
$PV = -1000$	$C/Y = 1$

FV gives 5237.38.

Remark: If we want to exclude the last payment, we subtract one PMT from FV:

$$\text{Value} = FV - PMT = 5237.38 - 300 = 4937.38$$

You may find that situation in some old past papers where payments were made at the beginning of each year. Since 2021, the IB has clarified that

- Payments (PMT) are made **at the end** of each year/period,
- FV is calculated **at the beginning** of the year/period (i.e. next day)

Thus, FV always includes the last payment!

EXAMPLE 6

An initial amount of 1000€ and then an extra amount of 1000€ at the end of each year are invested with an interest rate 12% compounded yearly. Find the value of the investment after 7 years.

By using GDC-Financial mode

$n = 7$	$PMT = -1000$
$I\% = 12$	$P/Y = 1$
$PV = -1000$	$C/Y = 1$

FV gives 12299.69.

EXAMPLE 7

An amount of 1000€ is invested with an interest rate 12% compounded **monthly**. An extra payment of 300€ is added at the **end of each year**. Find the value of the investment after 7 years.

By using GDC–Financial mode

$n = 7$	$PMT = -300$
$I\% = 12$	$P/Y = 1$
$PV = -1000$	$C/Y = 12$

FV gives 5397.73.

NOTICE.

If the regular payments take place yearly then

$$P/Y = 1$$

But if they are made in m periods per year, then

$$P/Y = m$$

(payments per year)

But now n is the total number of periods, i.e. $n = P/Y \times (\text{years})$.

EXAMPLE 8

An amount of 1000€ is invested with an interest rate 12% compounded **monthly**. An extra payment of 300€ is added at the **end of each month**. Find the value of the investment after 7 years.

By using GDC–Financial mode

$n = 7 \times 12 = 84$
$I\% = 12$
$PV = -1000$
$PMT = -300$
$P/Y = 12$
$C/Y = 12$

FV gives 41508.41.

♦ ANNUITY – AMORTIZATION

We invest PV but now, we regularly **withdraw** an amount PMT.

Notice for GDC–Financial mode

- We insert the payment PMT as a positive value [we receive]
- Here, the last payment is always included.

EXAMPLE 9

An amount of 1000€ euros is invested with an interest rate 12% compounded **monthly**. A **withdrawal** of 150€ is made at the end of each **year**. Find the value of the investment after 7 years.

By using GDC–Financial mode

$n = 7$
$I\% = 12$
$PV = -1000$
$PMT = 150$
$P/Y = 1$
$C/Y = 12$

FV gives 761.22.

In Example 9, the Future value after 1 year is

$$FV = 1000 \left(1 + \frac{12}{100 \times 12} \right)^{12} = 1126.83$$

So the annual interest is 126.83. Since our withdrawal of 150 is greater than this interest, FV will be zeroed out after a certain time. **Amortization** refers to the time needed for this to occur.

By using GDC–Financial mode in our example above:

Set FV = 0 and press n.

We obtain $n = 15.64$, so $n = 16$

For $n = 15$, the GDC gives $FV = 87.11$

For $n = 16$, the GDC gives $FV = -51.84$

Therefore, the amount left for the last withdrawal is

$$150 - 51.84 = 98.16 \text{ euros.}$$

♦ INFLATION – REAL VALUE OF THE INVESTMENT

In some problems the *inflation* is also taken into account. Apart from the interest rate $r\%$ they also give us the inflation rate $a\%$. Then

the future value (FV) of the investment
is translated into
the real value (RV) of the investment

Method 1 (the simplistic one):

We subtract the rates: $r' = r - a$

$$\text{Real value: } RV = PV \times \left(1 + \frac{r'}{100}\right)^n$$

But this is an approximation, (IB sometimes accepts it!!)

Method 2 (the correct one!):

- We find the future value as usual: $FV = PV \times \left(1 + \frac{r}{100}\right)^n$
- For the real value RV, we divide the FV by $\left(1 + \frac{a}{100}\right)^n$.

EXAMPLE 10

An amount of 1000€ euros is invested with

- an interest rate 5% per year ($r = 5\%$)
- an inflation rate 2% per year. ($a = 2\%$)

Find the *real value* of the investment after 10 years.

Method 1: $r' = 5\% - 2\% = 3\%$

$$\text{Real value: } RV = 1000 \times \left(1 + \frac{3}{100}\right)^{10} = 1343.92$$

Method 2: We find FV first

$$FV = 1000 \times \left(1 + \frac{5}{100}\right)^{10} = 1628.89$$

$$\text{Real value: } RV = \frac{FV}{\left(1 + \frac{2}{100}\right)^{10}} = \frac{1628.89}{(1.02)^{10}} = 1336.26$$

♦ EXPLANATION FOR PROBLEMS WITH PMT (mainly for HL)

You may ignore the remaining of this paragraph!

If the question involves payments (PMT), use the financial mode of GDC which yields immediate results as shown earlier. In the following 3 pages, we explain how FV is calculated in detail, for those who are interested.

If the annual interest rate is $r\%$, the ratio of the geom. sequence is:

$$R = 1 + \frac{r}{100}$$

The value of the investment after n years is given by

$$FV = \boxed{PV \times R^n} + \boxed{PMT \times \left(\frac{R^n - 1}{R - 1} \right)}$$

Indeed,

- PV is invested for n years: $\boxed{PV \left(1 + \frac{r}{100} \right)^n}$ [1st box]

- 1st PMT is invested for $n-1$ years:

- 2nd PMT is invested for $n-2$ years:

...

- last but one PMT is invested for 1 year:

- the last payment is just PMT

$$\begin{aligned} &PMT \left(1 + \frac{r}{100} \right)^{n-1} \\ &PMT \left(1 + \frac{r}{100} \right)^{n-2} \\ &\dots \\ &PMT \left(1 + \frac{r}{100} \right) \\ &PMT \end{aligned}$$

The sum of the last n terms is a G.S. with

$$u_1 = PMT, \text{ ratio: } R = 1 + \frac{r}{100}$$

Thus $S_n = \boxed{PMT \times \left(\frac{R^n - 1}{R - 1} \right)}$ [2nd box].

If each payment is equal to the present value ($PMT=PV$) we obtain in fact a G.S. of $n+1$ terms:

$$FV = \boxed{PV \times \left(\frac{R^{n+1} - 1}{R - 1} \right)}$$

Let us revisit the examples of this paragraph.

In EXAMPLE 5

$PV = 1000\text{€}$, $PMT = 300\text{€}$ at the *end* of each year
 $r=12\%$ compounded yearly (so $R=1.12$).

The value of the investment after 7 years is

$$FV = 1000 \times 1.12^7 + 300 \times \left(\frac{1.12^7 - 1}{1.12 - 1} \right) = 5237.38$$

In EXAMPLE 6

$PV = 1000\text{€}$, $PMT = 1000\text{€}$ at the *end* of each year
 $r=12\%$ compounded yearly (so $R=1.12$).

The value of the investment after 7 years is

$$FV = 1000 \times \left(\frac{1.12^8 - 1}{1.12 - 1} \right) = 12299.69$$

NOTICE. If

- the annual interest rate $r\%$ is compounded in k periods
- the payments occur yearly

then the ratio for 1 year is: $R = \left(1 + \frac{r}{100k} \right)^k$

In EXAMPLE 7

$PV = 1000\text{€}$, $PMT = 300\text{€}$ at the *end* of each year
 $r=12\%$ compounded **monthly**. Now the ratio for 1 year is

$$R = \left(1 + \frac{12}{100 \times 12} \right)^{12} = 1.01^{12}$$

The value of the investment after 7 years is

$$FV = 1000R^7 + 300 \times \left(\frac{R^7 - 1}{R - 1} \right) = 5397.73$$

NOTICE. If

- the annual interest rate $r\%$ is compounded in k periods
- the payments also occur in k periods

then the ratio for 1 period is: $R = \left(1 + \frac{r}{100k}\right)$

But now n is the total number of periods, i.e. $n = k \times (\text{years})$.

In EXAMPLE 8

$PV = 1000\text{€}$, $PMT = 300\text{€}$ at the end of each month.

$r = 12\%$ compounded monthly. Now the ratio for 1 month is

$$R = \left(1 + \frac{12}{100 \times 12}\right) = 1.01$$

The value of the investment after 7 years (so $n = 7 \times 12 = 84$) is

$$FV = 1000R^{84} + 300 \left(\frac{1.01^{84} - 1}{1.01 - 1} \right) = 41508.41$$

For amortization, the only difference is that we subtract the part of the payments.

$$FV = \boxed{PV \times R^n} - \boxed{PMT \times \left(\frac{R^n - 1}{R - 1} \right)}$$

In EXAMPLE 9

The ratio for 1 year is $R = \left(1 + \frac{12}{100 \times 12}\right)^{12} = 1.01^{12}$

The value of the investment after 7 years is

$$FV = 1000R^7 - 150 \left(\frac{R^7 - 1}{R - 1} \right) = 761.22$$

For Amortization we solve for n the equation $FV = 0$, i.e.

$$1000R^n = 150 \left(\frac{R^n - 1}{R - 1} \right)$$

The solution is 15.64 so the last withdrawal will take place after 16 years.